ON THE EXISTENCE OF MATRICES WITH PRESCRIBED HEIGHT AND LEVEL CHARACTERISTICS

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ABSTRACT

We determine all possible relations between the height (Weyr) characteristic and the level characteristic of an M-matrix. Under the assumption that the two characteristics have the same number of elements, we determine the possible relations between the two characteristics for a wider class of matrices, which also contains the class of strictly triangular matrices over an arbitrary field. Given two sequences which satisfy the above condition, we construct a loopless acyclic graph G with the following property: Every matrix whose graph is G has its height characteristic equal to the first sequence and its level characteristic equal to the second. We give several counterexamples to possible extensions of our results, and we raise some open problems.

1. Introduction

In this paper we determine all possible relations between the height (Weyr) characteristic and the level characteristic of an M-matrix. Indeed, under the assumption that the two characteristics have the same number of elements, we

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determine the possible relations between the two characteristics for a wider class of matrices, which also contains the class of strictly triangular matrices over an arbitrary field.

In the introduction to [15] it is observed that there is a relation between the two characteristics for an M-matrix, and in that paper and in several subsequent ones the case of equality was explored, e.g [13], [12], [1], [8], [9]. The question of characterizing all possible relations was thus raised implicitly in the 1950's, and the problem was stated explicitly in [16]. In this paper we solve this problem. Given two sequences η and λ of positive integers with the same number of elements, we show that there exists a matrix in the class considered with level characteristic λ and height characteristic η if and only if η majorizes λ when reordered in non-increasing order. This characterization was conjectured by Berman and van den Driessche [private communication, 1987].

We now describe our paper in more detail. Section 2 is devoted to definitions and notation. In particular we here define the level characteristic and the height characteristic of A. We also give our definition of majorization which is related to the definition found in many places, e.g. [11, p.7], but is not identical with it, since we wish to be consistent with our definition in [8].

In Section 3 we study the relation between the height and level characteristics for strictly lower triangular matrices under the assumption that the characteristics have the same number of elements. Given two sequences η and λ which satisfy the above condition, we construct a loopless acyclic graph G with the following property: Every matrix whose graph is G has its height characteristic equal to η and its level characteristic equal to λ. This result is then extended in Section 4 to the class of matrices all of whose singular vertices are simple (i.e. matrices which may be partitioned into a block triangular form, so that the singular diagonal blocks have 0 as a simple eigenvalue). This class, of course, contains the class of M-matrices. We thus obtain the results mentioned above. The paper is concluded with some partial results on the two characteristics for the class of matrices with all singular vertices simple, where we omit the assumption that the characteristics have the same number of elements.

In Sections 3 and 4 we give several examples which illustrate our theorems and we state counterexamples to possible extensions of our results. We also raise some open problems. For example, given sequences λ and η, it would be interesting to characterize all graphs G such that for all matrices A whose graph is G, the level characteristic of A equals λ and the height characteristic equals η.

This paper continues the series of joint papers [6], [7], [4], [5], [8], and [9], on the graph theoretic spectral theory of matrices. Other recent papers closely related to this series are [2] and [3]. These papers emphasize the relation between the combinatorial structure of a matrix and the structure of the generalised eigenspace of an eigenvalue of the matrix. Of particular interest in most of these papers is the case where the matrix is an M-matrix and the eigenvalue is 0.

2. Notation and Definitions

In this paper we assume that A is an $n \times n$ matrix over an arbitrary field. The index of A, that is the size of the largest Jordan block associated with 0 as an eigenvalue of A, is assumed to be p.

Notation 2.1: For a positive integer η we denote by $\langle n \rangle$ the set $\{1, \ldots, n\}$.

Notation 2.2: For a square matrix B we denote by $n(B)$ the nullity of B (the dimension of the nullspace of B), and by $r(B)$ the rank of B.

Notation 2.3: Let $\alpha \subseteq \langle n \rangle$. We denote by $A[\alpha]$ the principal submatrix of A whose rows and columns are indexed by α in the natural order.

Definition 2.4: For $i \in \langle p \rangle$ let $\eta_i(A) = n(A^i) - n(A^{i-1})$ (where $n(A^0) = 0$). The sequence $(\eta_1(A), \ldots, \eta_p(A))$ is called the **height characteristic** of A, and is denoted by $\eta(A)$. Normally we write η_i for $\eta_i(A)$ where no confusion should result.

We remark that the height characteristic of A is often referred to as the **Weyr characteristic** of A, e.g. [13].

We continue with some graph theoretic definitions. All the graphs we deal with are simple directed graphs.

Definition 2.5: The **graph** $G(A)$ of A is defined to be the graph with vertex set $\langle n \rangle$, and such that there is an arc from i to j if $a_{ij} \neq 0$.

Definition 2.6: Let G and H be two graphs with the same vertex set. We say that G is a **subgraph** of H, and we denote it by $G \subset H$, if every arc of G is an arc of H.

Notation 2.7: Let G be a graph. We denote the transitive closure of G by $\overline{G}$.

Definition 2.8: A graph is said to be **acyclic** if it contains no simple cycle other than loops. An acyclic graph is said to be **loopless** if it has no loops.

Definition 2.9: Let i be a vertex in an acyclic graph G. We define the **level** of i as the maximal length (number of vertices) of a simple chain in G that terminates at i. We call the set of all vertices of level j the j-th level of G, and we denote the cardinality of the j-th level of G by $\lambda_j(G)$. Let G have q levels. The sequence $(\lambda_1(G), \ldots, \lambda_q(G))$ is called the **level characteristic** of G, and is denoted by $\lambda(G)$.

We now assume that A is partitioned in a lower triangular $r \times r$ block form $(A_{ij})_1^r$, with square diagonal blocks. Note that the concepts defined in Definitions (2.10) - (2.14) below depend on the chosen partitioning for A, and that this is not explicitly noted there.

Definition 2.10: The **reduced graph** $R(A)$ of A is defined to be the graph with vertex set $\langle r \rangle$, and such that there is an arc from i to j if $A_{ij} \neq 0$. Note that since A is a lower triangular block matrix, $R(A)$ is acyclic.

Definition 2.11: (i) A vertex i in $R(A)$ is said to be **singular** if A_{ii} is singular. The set of all singular vertices of $R(A)$ is denoted by S.

(ii) A singular vertex i is said to be **simple** if 0 is a simple eigenvalue of A_{ii}.

Definition 2.12: The **singular graph** $S(A)$ of A is defined to be the graph with vertex set S, and such that there is an arc from i to j if $i = j$ or if there is a chain from i to j in $R(A)$. Note that $S(A)$ is a transitive acyclic graph.

Definition 2.13: The **level characteristic** $\lambda(A)$ of A is defined to be the sequence $\lambda(S(A))$.

Remark 2.14: After performing an identical permutation on the rows and the columns of A we may assume that A is in **Frobenius normal form**, namely a (lower) triangular block form, where the diagonal blocks are square irreducible matrices. As is well known, the Frobenius normal form is unique only up to certain permutations of the blocks and permutations within the blocks; see [13] and the references there for further information. However, the possible partitioning associated with the Frobenius normal forms of a given matrix A determine the same level characteristic for A.

Definition 2.15: A *Z-matrix* is a square matrix of the form $A = \alpha I - P$, where α is a real number and P is a (entrywise) nonnegative matrix. Such a *Z-matrix* is an *M-matrix* if α is greater than or equal to the spectral radius of P.

Convention 2.16: If A is an *M-matrix* or a triangular matrix, then we shall always assume that the partitioning of A which defines the reduced graph is the Frobenius normal form of A.

Observe that if A is a triangular matrix then $G(A) = R(A)$.

Notation 2.17: Let λ be a (finite) sequence of positive integers. We denote by $\hat{\lambda}$ the sequence λ reordered in a non-increasing order.

Definition 2.18: Let $\mu = (\mu_1, \dots, \mu_t)$ be a non-increasing sequence of positive integers. Consider the diagram formed by t columns of stars, such that the j-th column (from the left) has μ_j stars. The sequence μ^* *dual* to μ is defined as the sequence of row lengths of the diagram (read upwards).

We remark that a dual sequence is often called **a conjugate sequence**. Also, many equivalent definitions may be given for dual sequences, e.g. [9] and [11].

Definition 2.19: Let $\alpha = (\alpha_1, \dots, \alpha_t)$ and $\beta = (\beta_1, \dots, \beta_t)$ be sequences of non-negative integers. We say that β **majorizes** α, and denote it by $\alpha \preceq \beta$, if $\alpha_1 + \cdots + \alpha_k \leq \beta_1 + \cdots + \beta_k$ for every $k \in \langle t-1 \rangle$, and $\alpha + \cdots + \alpha = \beta_1 + \cdots + \beta_t$.

We remark that our definition is related to the definition of majorization as found in many places, e.g [11, p.7], but is not identical with it, since there β is said to majorize α if β reordered in a non-increasing order majorizes (in our sense) α reordered in a non-increasing order. Thus, the two definitions coincide if the sequences α and β are non-increasing.

3. The Existence of Triangular Matrices With Prescribed Characteristics

In this section we discuss strictly triangular matrices over an arbitrary field. We start with an easy proposition.

PROPOSITION 3.1: (i) *Let A be an $m \times m$ matrix such that $G \subseteq G(A) \subseteq \overline{G}$, where G is a graph that consists of a chain of length m. Then $\eta(A) = (1, \dots, 1)$.*
(ii) *Let $\eta = (\eta_1, \dots, \eta_p)$ be a non-increasing sequence of positive integers, and let G be a graph that consists of η_1 pairwise disjoint chains, where the sequence of lengths of the chains, ordered in a non-increasing order, is η^* . Then for every square matrix A with $G \subseteq G(A) \subseteq \overline{G}$ we have $\eta(A) = \eta$.*

Proof: . (i) Let $k \in \langle m-1 \rangle$. Observe that the k-th sub-diagonal of A^k is nonzero, and that the entries above it are all zero. Therefore, $r(A^k) = m - k$. Also, $A^m = 0$. The result follows.

(ii) Let $\eta^* = (\eta_1^*, \ldots, \eta_t^*)$, where $t = \eta_1$. Let A be a square matrix such that $G \subseteq G(A) \subseteq \bar{G}$. Then A is a direct sum of t matrices of sizes $\eta_1^*, \ldots, \eta_t^*$, each of which satisfies the conditions of part (i) of the proposition. Our claim follows from (i). □

The following lemma is immediate and well-known.

LEMMA 3.2: *Let A be a square matrix over an arbitrary field, let i be a vertex in $G(A)$, and let k be a positive integer. If there is no chain in $G(A)$ of length $k+1$ that starts at i, then the i-th row of A^k is a zero row.*

THEOREM 3.3: *Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a sequence of positive integers, and let $\eta = (\eta_1, \ldots, \eta_p)$ be a non-increasing sequence of positive integers. If $\hat{\lambda} \preceq \eta$ then there exists a loopless acyclic graph G such that for every matrix A over an arbitrary field with $G(A) = G$ we have $\lambda(A) = \lambda$ and $\eta(A) = \eta$. Furthermore, the graph G may be chosen to be transitive.*

Proof: Since $\hat{\lambda} \preceq \eta$, it follows that $\eta^* \preceq \hat{\lambda}^*$, see [11, p.174]. By the Gale-Ryser Theorem (e.g. [11, p.176]), there exists a $(0-1)$ $p \times \eta_1$ matrix E, where the sequence of the row sums of E, read from the bottom, is λ, and where the sequence of the column sums of E, read from the left, is η^*. Since η consists of p positive elements, it follows that the first element of η^* is p, so the first column of E consists of 1's. Let $n = \lambda_1 + \ldots + \lambda_p$. Then E has exactly n nonzero elements. We replace these nonzero elements in E by the numbers $1, \ldots, n$ (in any order), to obtain a new matrix F. We now construct a loopless acyclic graph G with vertex set $\langle n \rangle$ as follows. For every nonzero column of F let $(i_1, \ldots, i_t)$ be the nonzero elements of F in that column, read from the bottom. Then let G contain the chain $(i_1, \ldots, i_t)$. Also, for every $i \in \{2, \ldots, p\}$ let there be an arc from $f_{i,1}$ to every nonzero element in the $(i-1)$-th row of F. The latter assures that all the vertices in the i-th row of F are of level $\lambda_{p+1-i}, i \in \langle p \rangle$. Therefore, we have $\lambda(G) = \lambda$. Now let A be a matrix over an arbitrary field with $G(A) = G$. Clearly, $\lambda(A) = \lambda$. We claim that $\eta(A) = \eta$. Observe that after performing a permutation similarity on A, A may be partitioned as

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square. Furthermore, $A_{22} = A[\alpha]$ where α is the set of the p elements in the first column of F, and A_{11} is a direct sum of matrices $A[\beta_j]$ where β_j is the set of the nonzero elements in a nonzero column j of F, $j > 1$. Consider the matrix

$$A' = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

Observe that $\lambda(G(A')) = \eta$, and that $G(A')$ consists of η_1 pairwise disjoint chains, where the sequence of lengths of the chains, ordered in a non-increasing order, is η^* . By Proposition 3.1, we have $\eta(A') = \eta$. We now claim that $\eta(A) = \eta(A')$. We shall prove it by showing that for every $k \in \langle p \rangle$ we have $r(A^k) = r(A'^k)$. Clearly,

$$r(A^k) \geq r(A_{11}^k) + r(A_{22}^k) = r(A'^k). \tag{3.4}$$

Partition A^k conformably with the partitioning of A. Then

$$A^k = \begin{bmatrix} A_{11}^{\langle k \rangle} & 0 \\ A_{21}^{\langle k \rangle} & A_{22}^{\langle k \rangle} \end{bmatrix}$$

where $A_{ii}^{\langle k \rangle} = A_{ii}^k$, $i = 1, 2$. We have

$$r(A^k) \leq r(A_{11}^k) + r(\tilde{A}), \tag{3.5}$$

where $\tilde{A} = [A_{21}^{\langle k \rangle} A_{22}^{\langle k \rangle}]$. Observe that α contains exactly k elements of level greater than or equal to $p + 1 - k$. So, there is no chain of length $k + 1$ that starts at any of these elements, and by Lemma 3.2 the corresponding k rows of $\tilde{A}$ are zero rows. Therefore, we have $r(\tilde{A}) \leq p - k$. Since $r(\tilde{A}) \geq r(A_{22}^k) = p - k$, it now follows that $r(\tilde{A}) = r(A_{22}^k)$. By (3.4) and (3.5) we now obtain

$$r(A^k) = r(A_{11}^k) + r(A_{22}^k) = r(A'^k),$$

which proves our claim.

To see that the required graph may be chosen to be transitive, observe that we could apply the above proof to the transitive closure of G rather than to G.

□

We remark that in our report [10] we provide an example illustrating the proof of Theorem 3.3. Also, let G be the graph constructed in the proof of Theorem 3.3. In [10] we sketch an alternative proof to the fact that every matrix A with $G(A) = G$ satisfies $\eta(A) = \eta$, which does not use Proposition 3.1 or Lemma 3.2, and we provide there an example illustrating this alternative part of the proof of Theorem 3.3.

Remark 3.6: If there exists a loopless acyclic graph G such that for every matrix A over an arbitrary field with $G(A) = G$ we have $\lambda(A) = \lambda$ and $\eta(A) = \eta$, then λ and η must have the same number of elements. For proof, observe that we may choose all the nonzero elements of A to be negative, and hence A may be chosen to be an M-matrix. As is well known (e.g. [14]), the height and the level characteristics of an M-matrix consist of the same number of elements, which proves our claim. Therefore, Theorem 3.3 does not hold if we allow λ and η to have different number of elements (and if we replace $\hat{\lambda} \preceq \eta$ by $\hat{\lambda} \preceq (\eta, 0)$ as defined in Section 5).

In the proof of Theorem 3.3 we construct an acyclic graph G such that for every matrix A over an arbitrary field with $G(A) = G$ we have $\lambda(A) = \lambda$ and $\eta(A) = \eta$. Motivated by this we pose the following problem.

Problem 3.7: Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a sequence of positive integers, and let $\eta = (\eta_1, \ldots, \eta_p)$ be a non increasing sequence of positive integers, such that $\hat{\lambda} \preceq \eta$.

(i) Characterize all acyclic graphs G such that for every matrix A with $G(A) = G$ we have $\lambda(A) = \lambda$ and $\eta(A) = \eta$.

(ii) Characterize all acyclic graphs G such that for some matrix A with $G(A) = G$ we have $\lambda(A) = \lambda$ and $\eta(A) = \eta$.

Remark 3.8: Problems similar to Problems 3.7.(i), (ii) can be stated specifically for nonnegative matrices as well as M-matrices, where $G(A)$ is replaced by $S(A)$.

Remark 3.9: The case where $\lambda = \eta$ for M-matrices was discussed in great detail in [8] and [9], where 36 equivalent conditions are given.

4. Matrices With All Singular Vertices Simple

In this section we apply the result of the previous section in order to study the relations between the height characteristic and the level characteristic of a matrix A with all singular vertices simple. It follows from the Index Theorem proven in [2] as well as in [5], that the index p of A is less than or equal to the number q of levels in the singular graph of A. We start this section by proving a necessary and sufficient condition for given sequences λ and η to be the level and the height characteristics of a matrix satisfying certain conditions. In particular, our results apply to M-matrices, and hence solve a long standing problem, which was posed explicitly in [16] (see Remark 4.2).

THEOREM 4.1: *Let $\lambda = (\lambda_1, \ldots, \lambda_q)$ be a sequence of positive integers, and let $\eta = (\eta_1, \ldots, \eta_p)$ be a non-increasing sequence of positive integers. Then the following are equivalent.*

(i) *$p = q$, and $\hat{\lambda} \preceq \eta$.*

(ii) *$p = q$, and there exists a loopless transitive acyclic graph G such that for every matrix A over an arbitrary field with $G(A) = G$ we have $\lambda(A) = \lambda$ and $\eta(A) = \eta$.*

(iii) *$p = q$, and there exists a loopless acyclic graph G such that for every matrix A over an arbitrary field with $G(A) = G$ we have $\lambda(A) = \lambda$ and $\eta(A) = \eta$.*

(iv) *$p = q$, and there exists a strictly (lower) triangular matrix A such that $\lambda(A) = \lambda$ and $\eta(A) = \eta$.*

(v) *There exists a strictly (lower) triangular nonnegative matrix A such that $\lambda(A) = \lambda$ and $\eta(A) = \eta$.*

(vi) *There exists an M-matrix A such that $\lambda(A) = \lambda$ and $\eta(A) = \eta$.*

(vii) *$p = q$, and there exists a matrix A with all singular vertices simple such that $\lambda(A) = \lambda$ and $\eta(A) = \eta$.*

Proof: (i) $\Rightarrow$ (ii) follows from Theorem 3.3.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (vii) is immediate.

(iii) $\Rightarrow$ (v) is immediate.

(v) $\Rightarrow$ (vi) follows by taking the negative of a matrix satisfying (v).

(vi) $\Rightarrow$ (vii) is clear since M-matrices have all singular vertices simple.

(vii) $\Rightarrow$ (i). Let $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_p)$. By Theorem 3.5 in [3], (vii) implies $\hat{\lambda}_1 + \cdots + \hat{\lambda}_k \le \eta_1 + \cdots + \eta_k$, $k \in \langle p \rangle$. Since A has all singular vertices simple, it follows that $\hat{\lambda}_1 + \cdots + \hat{\lambda}_p = \eta_1 + \cdots + \eta_p$, and hence $\hat{\lambda} \preceq \eta$. □

Remark 4.2: Using the notation of [16], the equivalence of Conditions (i) and (vi) in Theorem 4.1 solves the $S_*(A)$ and $S_*(B)$ versions of Questions (8.8) and (8.9) in [16]. This result was conjectured by Berman and van den Driessche [private communication, 1987].

Remark 4.3: The implication (vii) $\Rightarrow$ (i) in Theorem 4.1 does not hold if A has a singular vertex which is not simple. Note that in such a case we have $\lambda_1 + \cdots + \lambda_q < \eta_1 + \cdots + \eta_p$, and hence we cannot have $\hat{\lambda} \preceq \eta$.

Remark 4.4: The following condition,

(a) *$p = q$, and for every loopless transitive acyclic graph G such that $\lambda(G) = \lambda$ there exists a matrix A with $G(A) = G$ such that $\lambda(A) = \lambda$ and $\eta(A) = \eta$,* immediately implies Condition (iv) in Theorem 4.1. However, Condition (a) is not implied by the conditions in Theorem 4.1. In fact, even the weaker condition, with $G(A)$ replaced by $R(A)$, is not implied by the conditions in Theorem 4.1 as demonstrated by the following example. Let $\lambda = \eta = (2,2)$, and let G be the loopless transitive acyclic graph

$$\begin{array}{ccc} 1 & & 2 \\ \uparrow & \nearrow & \\ 3 & & 4. \end{array}$$

Let A be a matrix with $R(A) = G$. If $\lambda(A) = \lambda$ then all the vertices in $R(A)$ are singular. If, furthermore, $\eta(A) = \eta$ then it follows that every vertex is a simple singular vertex. Since A is direct sum ($R(A)$ is not connected), it follows from Theorem 4.10 below that $\eta(A)$ is either (4,0) or (3,1), which is a contradiction. Therefore, there exists no matrix A with $R(A) = G$ such that $\lambda(A) = \lambda$ and $\eta(A) = \eta$.

We continue with three easy observations.

Observation 4.5: (i) Let λ be a sequences of p positive integers. Then the only sequence of p positive integers that majorizes $\hat{\lambda}$ is $\hat{\lambda}$ if and only if $\hat{\lambda}_2 = 1$.

(ii) Let η be a non-increasing sequences of p positive integers. Then the only non-increasing sequence of p positive integers that is majorized by η is η if and only if $\eta_1 \leq \eta_p + 1$.

(iii) Let η be a non-increasing sequences of p positive integers. Then every sequence α with $\hat{\alpha} = \eta$ satisfies $\alpha = \eta$ if and only if $\eta_p = \eta_1$.

In view of Observation 4.5, the following corollaries follow immediately from Theorem 4.1.

COROLLARY 4.6: *Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a sequence of positive integers. Then the following are equivalent.*

(i) *All M-matrices A with $\lambda(A) = \lambda$ have the same height characteristic.*

(ii) *All M-matrices A with $\hat{\lambda}(A) = \hat{\lambda}$ satisfy $\eta(A) = \hat{\lambda}$.*

(iii) $\hat{\lambda}_2 = 1$.

COROLLARY 4.7: *Let $\eta = (\eta_1, \ldots, \eta_p)$ be a non-increasing sequence of positive integers. Then the following are equivalent.*

(i) *All M-matrices A with $\eta(A) = \eta$ have the same reordered level characteristic $\hat{\lambda}(A)$.*

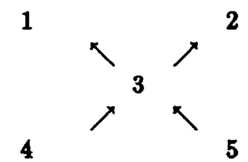
(ii) *All M-matrices A with $\eta(A) = \eta$ satisfy $\hat{\lambda}(A) = \eta$.*

(iii) $\eta_1 \leq \eta_p + 1$.

COROLLARY 4.8: *Let $\eta = (\eta_1, \ldots, \eta_p)$ be a non-increasing sequence of positive integers. Then the following are equivalent.*
(i) *All M-matrices A with $\eta(A) = \eta$ have the same level characteristic.*
(ii) *All M-matrices A with $\eta(A) = \eta$ satisfy $\lambda(A) = \eta$.*
(iii) $\eta_p = \eta_1$.

Let λ and η be sequences of p positive integers such that $\hat{\lambda} \preceq \eta$. Let G be a transitive graph such that $\lambda(G) = \lambda$. By the remark that follows Proposition (7.5) in [8], if there exists an M-matrix A with $R(A) = G$ such that $\lambda(A) = \lambda$ and $\eta(A) = \eta$, then there exists an M-matrix A with $G(A) = G$ such that $\lambda(A) = \lambda$ and $\eta(A) = \eta$. However, if G is not transitive then the above implication is false in general, as demonstrated by the following example.

Example 4.9: Let $\lambda = (2, 1, 2)$, and let G be the graph



satisfying $\lambda(G) = \lambda$. Now let $\eta = (2, 2, 1)$. Clearly, $\hat{\lambda} \preceq \eta$. The M-matrix

$$A = \left[\begin{array}{cccccccccc}
0 & \vdots & 0 & \vdots & 0 & 0 & \vdots & 0 & \vdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \vdots & 0 & \vdots & 0 & 0 & \vdots & 0 & \vdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-1 & \vdots & 0 & \vdots & 1 & -1 & \vdots & 0 & \vdots & 0 \\
 & & & & & & & & & \\
0 & \vdots & -1 & \vdots & -1 & 1 & \vdots & 0 & \vdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \vdots & 0 & \vdots & -1 & 0 & \vdots & 0 & \vdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \vdots & 0 & \vdots & 0 & -1 & \vdots & 0 & \vdots & 0
\end{array}\right]$$

satisfies $R(A) = G$, and we have $\lambda(A) = \lambda$ and $\eta(A) = \eta$. However, it is easy to verify that every matrix A with $G(A) = G$ satisfies $\eta(A) = (3, 1, 1)$, and so $\eta(A) \neq \eta$.

So far in this section, we have discussed the case of matrices A with all singular vertices simple, for which we have $p = q$. We conclude the paper with a brief discussion of the case $p < q$. We remark that the partitioning of A is not necessarily the Frobenius normal form.

Let $\lambda = (\lambda_1, \dots, \lambda_q)$ and $\eta = (\eta_1, \dots, \eta_p)$, where $p \le q$. We use the notation $(\eta, 0)$ for the sequence $\alpha = (\alpha_1, \dots, \alpha_q)$, where $\alpha_i = \eta_i$ for $i \le p$ and $\alpha_i = 0$ for $p < i \le q$. The following theorem follows from Theorem (3.5) in [3] and the remark that follows there.

THEOREM 4.10: *Let A have all singular vertices simple. Then $\hat{\lambda}(A) \preceq (\eta(A), 0)$.*

Motivated by Theorem (4.10), we ask the following.

Question 4.11: Let $\lambda = (\lambda_1, \dots, \lambda_q)$ be a sequence of positive integers, and let $\eta = (\eta_1, \dots, \eta_p)$ be a non-increasing sequence of positive integers. Assume further that $p \le q$ and that $\hat{\lambda} \preceq (\eta, 0)$. Does there exist a matrix A with $\lambda(A) = \lambda$ and $\eta(A) = \eta$?

A special case of Theorem 4.10 is the following.

THEOREM 4.12: *Let G be a loopless acyclic graph with $\lambda(G) = \lambda = (\lambda_1, \dots, \lambda$ Then for every matrix A with $G(A) = G$ we have $\hat{\lambda} \preceq (\eta(A), 0)$.*

Here too, one might ask a question similar to Question 4.11. That is, given a sequence $\lambda = (\lambda_1, \dots, \lambda_q)$ of positive integers, and a non-increasing sequence $\eta = (\eta_1, \dots, \eta_p)$ of positive integers, such that $p \le q$ and $\hat{\lambda} \preceq (\eta, 0)$. Does there exist a loopless acyclic graph G with $\lambda(G) = \lambda$ and a matrix A with $G(A) = G$ such that $\eta(A) = \eta$? However, the answer to this question is negative, as demonstrated by choosing $\lambda = (1, 1)$ and $\eta = (2)$. Clearly, $\hat{\lambda} \preceq (\eta, 0)$. However, for every loopless acyclic graph G with $\lambda(G) = \lambda$ and every matrix A with $G(A) = G$, we have $\eta(A) = (1, 1)$.

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